MA3 – RECITATION QUESTIONS

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1. Week 1

Example 1.1 (Permutation)**.** How many different 7-place Californian license plates are possible? (1-number, 3 consecutive letters and 3 consecutive numbers)

Example 1.2 (Dividing into distinctive groups)**.**
(1) How many possible ways to arrange the word "Wollongong"?
(2) The game of bridge is played by 4 players, each of whom is dealt 13 cards. How many bridge deals are possible?
(3) What is the coefficient of $w^2x^3yz^3$ in the expansion of $(w+x+y+z)^9$?

Example 1.3 ("Balls in urns")**.** Nine students, five men and four women, interview for four summer internships sponsored by a city newspaper.

(1) In how many ways can the newspaper choose a set of four interns?
(2) In how many ways can the newspaper choose a set of four interns if it must include two men and two women in each set?
(3) What is the probability that out of 4 interns that have been chosen randomly, not everyone is of the same sex?

Example 1.4 (Placement problem)**.** There are 5 seats in a row in a movie theater. You know that two of your friends just had a fight and they do not want to sit together for the duration of the movie. How many possible ways can you assign seats for your 5 friends?

Example 1.5 (Cards problems)**.** In a standard deck of 52 cards (perfectly shuffled)

(1) what is the probability that the k-th card is an queen?
(2) what is the probability that the first queen appears in the k-th place, where $k \leq 52$?
(3) a bridge hand (thirteen cards) is dealt. Let A be the event that the hand contains four aces; let B be the event that the hand contains four kings. Find $P(A \cup B)$.
(4) two cards are distributed to each of three players. What is the probability that at most one player has one ace and one king? (hint: exclusion-inclusion principle)

Example 1.6 (Dice problem)**.**
(1) Five fair dice are rolled. What is the probability that the faces showing constitute a "full house"—that is, three faces show one number and two faces show a second number?

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(2) Roll a fair dice until a 5 or 6 comes up. What is the probability that we see 5 before 6?

2. Week 2

Example 2.1. Show that

(1)
$$\mathbb{P}[A \cap (B \cap C)^c] = \mathbb{P}[A] - \mathbb{P}[A \cap B \cap C]$$

(2)
$$\mathbb{P}[A \cap B \cap C] = \mathbb{P}[A|B \cap C] \cdot \mathbb{P}[B|C] \cdot \mathbb{P}[C]$$

(3) if $\mathbb{P}[A] > 0$, then
$$\mathbb{P}[A \cap B|A] \geq \mathbb{P}[A \cap B|A \cup B].$$

(4)
$$\mathbb{P}[A|B] = \frac{\mathbb{P}[B|A]\mathbb{P}[A]}{\mathbb{P}[B|A]\mathbb{P}[A] + \mathbb{P}[B|A^c]\mathbb{P}[A^c]}.$$

(5) if A and B are independent, then A and B^c are also independent.

Example 2.2. A man has n keys on a key ring, one of which opens the door to his apartment. He asked one of his friends to pick up something from his apartment without telling his friend which key was the apartment key. His friend came up with a plan. He will choose a key at random and try it. If it fails to open the door, he will discard it and choose one of the remaining $n-1$ keys randomly. Show that the probability that the door opens with the third key he tries equals that of the first key.

Example 2.3. Alice and Bob are playing a game where they both roll a die and the higher roll wins.

(1) Alice rolls a 4. What is the probability she wins?
(2) What is the probability that someone wins?
(3) Is the probability that someone wins independent of
   (a) what Alice/Bob rolls?
   (b) of both their rolls?
   (c) of the sum of their rolls?
(4) Suppose Alice won. What is the probability she rolled a 4?

Example 2.4. An urn contains w white chips, b black chips, and r red chips. The chips are drawn out at random, one at a time, with replacement. What is the probability that a white appears before a red?

3. Week 3

Example 3.1. Let n be a positive integer, and let X be a random variable which is uniformly distributed among the numbers $\{0, 1, 2, \ldots, 2n\}$. Compute $\mathbb{E}(X)$ and $Var(X)$.

Example 3.2. Let $X \sim Poisson(\lambda)$ for some $\lambda > 0$. For any given $t \in \mathbb{R}$, compute the moment generating function of X, $\mathbb{E}[e^{tX}]$.

Example 3.3. Suppose a coin turns heads with probability $p \in (0, 1)$. What is the expected number of coin flips until we first get heads.

Example 3.4. If a typist averages one misspelling in every 3250 words, what are the chances that a 6000-word report is free of all such errors? (try both Binomial analysis and Poisson approximation)

4. Week 4-5

Example 4.1. The number of miles a Tesla can run before dying out its battery is given an exponential distribution. The average number of miles is about $300,000$. What is the probability that you can drive for $150,000$ without replacing its battery?

Example 4.2. Let $\Phi(\cdot)$ be the CDF of $X \sim \mathcal{N}(0, 1)$ (standard normal distribution). What is the expected value and variance of $X = \Phi^{-1}(U)$ where $U \sim \mathcal{U}[0, 1]$ (uniform on the unit interval)?

Example 4.3. Suppose that two fair dice are tossed one time. Let X denote the number of 2's that appear, and Y the number of 3's. Write the matrix giving the joint probability density function for X and Y . Suppose a third random variable, Z, is defined, where $Z = X + Y$. Use $p_{X,Y}(x, y)$ to find $p_Z(z)$, and use $p_Z(z)$ to find F_Z.

Example 4.4. Let X and Y be two continuous and independent random variable with marginal distributions $f_X(x) = x, 0 \leq x \leq 1$ and $f_Y(y) = 1, 0 \leq y \leq 1$. Find $\mathbb{P}[\frac{Y}{X} > 2]$.

Example 4.5. Let Y be a non-negative, continuous random variable. Show that $W = Y^2$ has pdf $f_W(w) = \frac{1}{2\sqrt{w}} f_Y(\sqrt{w})$.

Example 4.6. Consider two random variables X and Y, which are joint uniformly distributed on the unit square (i.e., $f_{X,Y}(x, y) = 1$) on $x, y \in [0, 1]$. Compute the following:

(1) $\mathbb{E}[X]$
(2) $\mathbb{E}[X|Y]$
(3) $\mathbb{E}[X|Y > X]$

5. Week 6

Example 5.1. Suppose X_1, X_2, X_3, X_4 are uncorrelated random variables (i.e., $Cov(X_i, X_j) = 0$ for $i \neq j$) where each X_i has an expectation of 0 and variance of 1. What is the correlation between $X_1 + X_2$ and $X_2 + X_3$? What about the correlation between $X_1 + X_2$ and $X_3 + X_4$?

Example 5.2. Let Y_1, Y_2, Y_3 be i.i.d. Exponential with $\lambda = 1$. Let $Y_{min} = \min\{Y_1, Y_2, Y_3\}$. Compare $f_{Y_{min}}(y)$ with $f_{Y_1}(y)$ and compute $\mathbb{P}[Y_1 < 1]$ and $\mathbb{P}[Y_{min} < 1]$ (intuitively think about which one should be larger).

Example 5.3. Given the joint pdf

$$f_{X,Y}(x,y) = 2 \cdot e^{-(x+y)}, 0 \leq x \leq y, y \geq 0,$$

find

(1) $\mathbb{P}[Y < 1 | X < 1]$
(2) $\mathbb{P}[Y < 1 | X = 1]$
(3) $f_{Y|X}(y)$
(4) $\mathbb{E}[Y|X]$
(5) $Var[Y|X]$

6. Week 7

Example 6.1. A factory produces microwaves. Suppose each microwave works with probability p and is defective with probability $1-p$. Whether each microwave is defective is independent of the others. You test n randomly selected microwaves. Let $X_i \sim Bernoulli(p)$ be the indicator for i-th microwave being defective.

(1) Does $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ give us a good way to estimate p?
(2) If you test 100 microwaves, what does Chebyshev's inequality tell us about the probability that $\bar{X}_{100}$ falls in the range $(p - 0.1, p + 0.1)$?
(3) How many microwaves should we test to ensure a 99% probability of measuring $\bar{X}_n$ in $(p - 0.1, p + 0.1)$?

Example 6.2. A factory produces X_n microwaves on day n, where $X_1, X_2, \ldots$ are iid random variables with mean 20 and variance 5.

(1) Approximate the probability that the number of microwaves produced in 100 days is less than 1950.
(2) Let N be the first day that the total number of microwaves produced exceeds 10,000. Calculate the approximated probability that $N \geq 505$.

7. Week 8

Example 7.1. Let $X_1, \ldots, X_5 \sim \mathcal{N}(\mu, \sigma^2 = 4)$ where μ is unknown. We set the significance level $\alpha = 0.05$ and your hypotheses are $H_0 : \mu = 77$ and $H_1 : \mu < 77$. Suppose the sample mean that you observe is $\bar{X}_5 = 76.1$. Do you reject the null hypothesis based on these observations?

Example 7.2. We roll a weighted dice and get $3, 3, 3, 4, 1, 1, 6, 2, 6, 2$.

(1) Not knowing anything about the probabilities, estimate the mean and variance.
(2) What happens if we know that opposite faces, e.g., $(1, 6), (2, 5), (4, 3)$ have the same probabilities?

Example 7.3. Let X be a discrete random variable with p.m.f.

$$X = \begin{cases} 0 & p_\theta(x) = \frac{2\theta}{3} \\ 1 & p_\theta(x) = \frac{\theta}{3} \\ 2 & p_\theta(x) = \frac{2(1-\theta)}{3} \\ 3 & p_\theta(x) = \frac{(1-\theta)}{3} \end{cases}$$

We draw 10 independent observations from this distribution: $3, 0, 2, 1, 3, 2, 1, 0, 2, 1$.

(1) Find $\hat{\theta}_{MLE}$, the maximum likelihood estimator of θ.

(2) Find $\hat{\theta}_{MOM}$, the method of moments estimator of θ.